Entry Task:

Find all critical points of

$$z = f(x,y) = 2x^4 + y^2 - 4xy + 1$$

$$\frac{\partial z}{\partial x} = 8x^3 - 4y \overset{?}{=} 0 \Rightarrow 8x^3 = 4y \Rightarrow y = 2x^3$$

$$\frac{\partial z}{\partial y} = 2y - 4x \overset{?}{=} 0$$

COMBINE!!!

$$2(2x^3) - 4x \overset{?}{=} 0$$
$$4x^3 - 4x \overset{?}{=} 0$$
$$4x(x^2-1) \overset{?}{=} 0$$
$$x = 0 \quad \text{or} \quad x^2 = 1$$
$$x = -1 \qquad x = 1$$

THREE POSSIBILITIES

$$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$$
$$x = -1 \Rightarrow y = -2 \Rightarrow (-1,-2)$$
$$x = 1 \Rightarrow y = 2 \Rightarrow (1,2)$$

You do (HW Problem 14.2/5)

$$z = -6x^2 + 2x - 4y^2 - 3y + 8xy + 30$$

$= f(x,y)$

a. Find the critical point.

$\frac{\partial z}{\partial x} = -12x + 2 + 8y \overset{?}{=} 0$

$8y = 12x - 2$

$y = \frac{12x-2}{8}$

$\frac{\partial z}{\partial y} = -8y - 3 + 8x \overset{?}{=} 0$

COMBINE!! $-8\left(\frac{12x-2}{8}\right) - 3 + 8x \overset{?}{=} 0$

$-12x + 2 - 3 + 8x = 0$

$-1 = 4x$

$\boxed{x = -\frac{1}{4} \Rightarrow y = \frac{-3-2}{8} = -\frac{5}{8}}$

b. Find the largest and smallest values of f(2,y) on the interval y=-4 to y=0.

$f(2,y) = -24 + 4 - 4y^2 - 3y + 16y + 30$

$= -4y^2 + 13y + 10$

$f_y(2,y) = -8y + 13 = 0 \Rightarrow y = \frac{13}{8} = 1.625$

OUTSIDE RANGE

$f(2,-4) = -4(-4)^2 + 13(-4) + 10 = \boxed{-106}$

$f(2,0) = -4(0) + 13(0) + 10 = \boxed{10}$

c. Suppose (x,y) = (-6,-7).

A small increase in x will lead to a LARGER/SMALLER increase in z than a small increase in y. (circle either larger or smaller)

$\frac{\partial z}{\partial x}$

$f_x(-6,-7) = -12(-6) + 2 + 8(-7)$

$= 18$

$f_y(-6,-7) = -8(-7) - 3 + 8(-6)$

$= 5$

d. Which is steepest at x = 1?

f(x,4), f(x,6), f(x,8) or f(x,10)

$f(x,4) = -6x^2 + 2x - 64 - 12 + 32x + 30$

$f_x(1,4) = -12(1) + 2 + 8(4) = 22$

$f_x(1,6) = -12(1) + 2 + 8(6) = 38$

$f_x(1,8) = -12(1) + 2 + 8(8) = 54$

$f_x(1,10) = -12(1) + 2 + 8(10) = 70 \Leftarrow$

Example (From HW 14.2/6)

Find the critical point for

$$f(x,y) = 12 + xy + \frac{27}{x} + \frac{8}{y} = 12 + xy + 27x^{-1} + 8y^{-1}$$

$$f_x = y - 27x^{-2} = y - \frac{27}{x^2} \stackrel{?}{=} 0$$

$$y = \frac{27}{x^2}$$

COMBINE !!!

$$f_y = x - 8y^{-2} = \underbrace{x - \frac{8}{y^2}}_{} \stackrel{?}{=} 0 \quad \cdot y^2$$

$$xy^2 - 8 = 0$$

$$x\left(\frac{27}{x^2}\right)^2 - 8 \stackrel{?}{=} 0$$

$$x \cdot \frac{729}{x^4} - 8 = 0$$

$$\frac{729}{x^3} - 8 = 0$$

$$\Rightarrow 729 - 8x^3 = 0$$

$$\Rightarrow 729 = 8x^3$$

$$\Rightarrow x^3 = \frac{729}{8}$$

$$\Rightarrow x = \left(\frac{729}{8}\right)^{1/3}$$

$$\boxed{x = 4.5}$$

$$\boxed{y = \frac{27}{(4.5)^2} = 1.\overline{3}}$$

More Applications

Cost Breakdown (14.3/1-2)

Suppose the cost to produce ONE item is given by:

$$C(x, y) = 3x^2 + 4y^2 + 5xy + 10,$$

where

$x =$ cost for 1 hour of labor, and

$y =$ cost for 1 pound of materials.

$$C_x = 6x + 5y \qquad \frac{\text{cost}}{\text{labor hours}}$$

$$C_y = 8y + 5x \qquad \frac{\text{cost}}{\text{pound}}$$

Question:

The current hourly rate for labor is \$20 and material is \$55 per pound.

How will a \$1 per hour raise for labor affect the cost to produce 1 item?

$$C_x(20, 55) = 6(20) + 5(55)$$

$$= \boxed{395}$$

Marginal Productivity (14.3/5-6)

Suppose that the number of crates of a particular fruit produced is

$$z = \frac{9xy - 0.0002x^2 - 5y}{0.03x + 4y} \begin{array}{l} \leftarrow N \\ \leftarrow D \end{array}$$

where

$x =$ number of hours of labor, and

$y =$ number of acres of the crop.

Find the marginal productivity of the number of hours of labor when x = 100 and y = 200.

Interpret your answer

$$\frac{\partial z}{\partial x} = \frac{\text{crates}}{\text{labor hours}} = \text{marginal productivity}$$

$$N = 9xy - 0.0002x^2 - 5y$$

$$N' = 9y - 0.0004x$$

$$D = 0.03x + 4y$$

$$D' = 0.03$$

$$\frac{\partial z}{\partial x} = \frac{(0.03x+4y)(9y-0.0004x) - (9xy-0.0002x^2-5y)(0.03)}{(0.03x+4y)^2}$$

at $x = 100,\ y = 200$

$$\frac{\partial z}{\partial x} = \frac{(3+800)(1800-0.04) - (9(100)(200) - 0.0002(100)^2 - 5(200))(0.03)}{(3+800)^2}$$

$$\approx 2.23322 \quad \frac{\text{crates}}{\text{labor hours}}$$